Homework #2

Problem 1)
Part a)

\[ T = \frac{Q}{MC} \cdot t + \text{constant} \]

Plot using Excel, MATLAB, or another program that can do linear regression

Using Excel...

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>500</td>
<td>31</td>
</tr>
<tr>
<td>1000</td>
<td>37</td>
</tr>
<tr>
<td>1500</td>
<td>43</td>
</tr>
</tbody>
</table>

Beaker of Water Heated with 110 Watts

\[ T = 0.0126 \cdot t + 24.3 \]

\[ R^2 = 0.9985 \]

\[ \frac{Q}{MC} = 0.0126 \]

\[ MC = \frac{Q}{0.0126} = 8730.16 \frac{J}{K} \]

\[ SSE = \sum_{i=1}^{n} (T_i - f(t_i))^2 = 0.3 \]
Part b)

\[ MC \frac{dT}{dt} = -UA(T - T_a) \]

\[ \frac{dT}{(T - T_a)} = - \frac{UA}{MC} dt \]

\[ \ln(T - T_a) = - \frac{UA}{MC} t + c_1 \]

\[ T = c_1 e^{-\frac{UA}{MC} t} + T_a \]

Many ways to solve...
- Using an exponential fit in Excel or MATLAB
- Rearranging equation to create a linear model (shown below)

**Plot \( \ln(T-T_a) \) versus time**

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>( \ln(T-T_a) )</th>
<th>Trendline</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.995732274</td>
<td>2.944166</td>
</tr>
<tr>
<td>5000</td>
<td>2.63905733</td>
<td>2.659188</td>
</tr>
<tr>
<td>15000</td>
<td>2.197224577</td>
<td>2.089232</td>
</tr>
<tr>
<td>20000</td>
<td>1.791759469</td>
<td>1.804254</td>
</tr>
<tr>
<td>Slope</td>
<td>-5.69956E-05</td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>2.944165537</td>
<td></td>
</tr>
</tbody>
</table>

![Fluid Cooling](image)

\[- \frac{UA}{MC} = -5.69956E^{-5}\]

\[ UA = 5.69956E^{-5} \cdot 8730.16 = 0.4976 \text{ W/K} \]

\[ SSE = 1.851 \]
Part c) Solve differential equation to obtain the function $T(t)$:

$$MC \frac{dT}{dt} = Q - UA(T - T_a)$$

Rearrange and solve, I chose to use the method below

$$\frac{dT_h}{dt} + \frac{UA}{MC} T = 0$$

$$T_h = a_1 e^{-\frac{UA}{MC} t}$$

$$T_p = a_1 e^{-\frac{UA}{MC} t} \cdot \int \left( \frac{Q + UA T_a}{MC a_1 e^{-\frac{UA}{MC} t}} \right) dt$$

$$T_p = e^{-\frac{UA}{MC} t} \cdot \left( \frac{Q + UA T_a}{UA} \cdot \frac{UA}{e^{\frac{UA}{MC} t}} + a_2 \right)$$

$$T(t) = a_2 e^{-\frac{UA}{MC} t} + \frac{Q + UA T_a}{UA}$$

$$T(0) = 24^\circ C = a_2 + \frac{Q + UA T_a}{UA}
\quad a_2 = 24 - \frac{Q + UA T_a}{UA} = 24 - \frac{119 + 0.4976 \cdot 25}{0.4976} = -220.0611$$

$$T = 246.0611 - 220.0611 \cdot e^{-0.0000569978}$$

$$SSE = 1.71$$

HOWEVER! We are using $MC$ calculated from a straight line fit and $UA$ that is calculated using that $MC$. If the model really should include heat loss then $MC$ and $UA$ should be calculated by trying to minimize the $SSE$ of the two equations below.

$$T = \frac{Q + UA T_a}{UA} + \left( T_o - \frac{Q + UA T_a}{UA} \right) \cdot e^{-\frac{UA}{MC} t}$$

$$T = c_1 e^{-\frac{UA}{MC} t} + T_o$$

Below I’ve minimized the two $SSE$s and it is clearly lower than the $SSE$s obtained using the $MC$ and $UA$ obtained in parts a and b

<table>
<thead>
<tr>
<th>Time</th>
<th>Temperature</th>
<th>$T_{\text{calc}}$</th>
<th>$(T-T_{\text{calc}})^2$</th>
<th>Time</th>
<th>Temperature</th>
<th>$T_{\text{calc}}$</th>
<th>$(T-T_{\text{calc}})^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(seconds)</td>
<td>(°C)</td>
<td>(°C)</td>
<td></td>
<td>(seconds)</td>
<td>(°C)</td>
<td>(°C)</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>24</td>
<td>24</td>
<td>0</td>
<td>45</td>
<td>45</td>
<td>45</td>
<td>0</td>
</tr>
<tr>
<td>500</td>
<td>31</td>
<td>30.56653</td>
<td>0.1879</td>
<td>5000</td>
<td>39</td>
<td>40.01213</td>
<td>1.024399</td>
</tr>
<tr>
<td>1000</td>
<td>37</td>
<td>36.95738</td>
<td>0.001817</td>
<td>15000</td>
<td>34</td>
<td>33.30912</td>
<td>0.477322</td>
</tr>
<tr>
<td>1500</td>
<td>43</td>
<td>43.17725</td>
<td>0.031418</td>
<td>20000</td>
<td>31</td>
<td>31.09803</td>
<td>0.009611</td>
</tr>
<tr>
<td></td>
<td>SSE</td>
<td>0.221134</td>
<td></td>
<td></td>
<td>SSE</td>
<td>SSE</td>
<td>1.511332</td>
</tr>
<tr>
<td>UA</td>
<td>0.449993539</td>
<td></td>
<td></td>
<td>MC</td>
<td>8297.071472</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The table and graphs below show the comparisons of the answers. Modeling the heat loss can benefit the model but not to a significant degree in this regime. More work is also required because new UA's and MC's would be required to make the heat loss included model beneficial.

<table>
<thead>
<tr>
<th></th>
<th>Heating</th>
<th></th>
<th>Cooling</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Linear</td>
<td>Heat loss</td>
<td>Optimized</td>
<td>Part B</td>
</tr>
<tr>
<td>UA (W/K)</td>
<td>0.4976</td>
<td>0.44999</td>
<td>0.4976</td>
<td>0.44999</td>
</tr>
<tr>
<td>MC (J/K)</td>
<td>8730.16</td>
<td>8730.16</td>
<td>8297.07</td>
<td>8730.16</td>
</tr>
<tr>
<td>SSE</td>
<td>0.3</td>
<td>1.71</td>
<td>0.22</td>
<td>1.85</td>
</tr>
</tbody>
</table>

**Beaker of Water Heated with 110 Watts**

Heating of the beaker
Beaker of Water Cooling

Cooling of the beaker
Problem 2) From slide 28 (chapter 2)

\[
\frac{dy}{dt} = -y + 6e^{-5t}
\]

\[
y = \frac{T - \overline{T}}{W_d - \overline{W}_a}, \quad \overline{T} = 140^\circ C, \quad \overline{W}_a = 0.83e^{-\frac{t}{10}}
\]

\[
y(0) = 50
\]

\[
2s y(s) - 50 = -\frac{y(s)}{s}
\]

\[
y(s)[2s + 1] = \frac{6e^{-5s} \cdot 50}{s}
\]

\[
y(s) = \frac{6e^{-5s} \cdot 50}{s(2s + 1)} = \frac{(0.5)(6e^{-5s})}{s(2s + 1)} + \frac{50}{2s + 1}
\]

\[
y(t) = (6e^{-5t})(1 - e^{-0.5t}) + 50e^{-0.5t}
\]

\[
T - \overline{T} = (6e^{-5t})(W_d - \overline{W}_a)(1 - e^{-0.5t}) + 50e^{-0.5t}
\]

\[
0 = 6e^{-5t}(0 - 0.83e^{-t})(1 - e^{-0.5t}) + 50e^{-0.5t}
\]

Solve for \( t \)

\[
t = 1.39 \text{ hours}
\]
Problem 3) Combine 2 first order ODE into one second order ODE by eliminating $T_e$

$$wC(T_i + h_e A(T_e - T)) - wC = mc \frac{dT}{dt}$$  \hspace{1cm} (1)

$$Q - h_e A(T_e - T) = m c_e \frac{dT_e}{dt}$$  \hspace{1cm} (2)

First solve (1) for $T_e$, then differentiate to find $\frac{dT_e}{dt}$.

Substitute $T_e$ into (2)

$$wC(T_i + h_e A(T_e - T)) - wC = mc \frac{dT}{dt}$$

$$T_e = \frac{mc \frac{dT}{dt} + wC - wC(T_i + h_e A T)}{h_e A}$$

$$\frac{dT_e}{dt} = \frac{mc \frac{dT}{dt}}{h_e A} + \frac{dT}{dt} \left( \frac{wc}{h_e A} + 1 \right) - \frac{dT_i}{dt} \frac{wc}{h_e A}$$

Plug into 2

$$Q = h_e A \left( mc \frac{dT}{dt} + wC - wC(T_i + h_e A T) \right) + h_e A T = m c_e \left( \frac{mc \frac{dT}{dt}}{h_e A} + \frac{dT}{dt} \left( \frac{wc}{h_e A} + 1 \right) - \frac{dT_i}{dt} \frac{wc}{h_e A} \right)$$

$$mc c_m \frac{dT}{dt} + \frac{dT}{dt} \left( \frac{mc c_e}{h_e A} + \frac{mc c_e}{wC - wC} \right) = \frac{dT_i}{dt} wC - dC + \frac{dT_i}{dt} wC + \frac{Q}{wC}$$
Problem 3.6)

a) $Y(s) = \frac{s(s+1)}{(s+2)(s+3)(s+4)} = \frac{\alpha_1}{s+2} + \frac{\alpha_2}{s+3} + \frac{\alpha_3}{s+4}$

$x_1 = \frac{s(s+1)}{(s+3)(s+4)} \bigg|_{s=-2}$

$x_2 = \frac{s(s+1)}{(s+2)(s+4)} \bigg|_{s=-3}$

$x_3 = \frac{s(s+1)}{(s+2)(s+3)} \bigg|_{s=-4}$

$x(s) = \frac{1}{s+2} - \frac{1}{s+3} + \frac{1}{s+4}$

$x(t) = e^{-2t} - e^{-3t} + e^{-4t}$

b) $Y(s) = \frac{s+1}{(s+1)^2} = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{(s+1)^2}$

$x_1 = \frac{s+1}{(s+1)^2} \bigg|_{s=-1}$

$x_2 = 3$

$x(s) = \frac{1}{s+1} + \frac{3}{(s+1)^2}$

$x(t) = e^{-t} + 3te^{-t}$

c) $Y(s) = \frac{1}{s^2 + 4}$

$x(s) = \frac{1}{s^2 + (2s+1)} = \frac{1}{(s+1)^2 + 0.75} = \frac{1}{(s+1)^2 + 0.5^2}$

$x(t) = \frac{1}{1.155} e^{-1.5t} \sin(0.8466t)$
3.9

a) \[ X(s) = \frac{6(s+2)}{(s^2 + 9s + 20)(s+4)} = \frac{6(s+2)}{(s+4)(s+5)(s+4)} \]

\[ x(0) = \lim_{s \to \infty} \left[ \frac{6s(s+2)}{(s+5)(s+4)^2} \right] = 0 \]

\[ x(\infty) = \lim_{s \to 0} \left[ \frac{6s(s+2)}{(s+5)(s+4)^2} \right] = 0 \]

\( x(t) \) is converging (or bounded) because \([sX(s)]\) does not have a limit at \( s = -4 \) and \( s = -5 \) only, i.e., it has a limit for all real values of \( s \geq 0 \).

\( x(t) \) is smooth because the denominator of \([sX(s)]\) is a product of real factors only. See Fig. S3.9a.

**Figure S3.9a. Simulation of \( X(s) \) for case a)**
b) \[ X(s) = \frac{10s^2 - 3}{(s^2 - 6s + 10)(s + 2)} = \frac{10s^2 - 3}{(s - 3 + 2j)(s - 3 - 2j)(s + 2)} \]

\[ x(0) = \lim_{s \to \infty} \left[ \frac{10s^3 - 3s}{(s^2 - 6s + 10)(s + 2)} \right] = 10 \]

3-9

Application of final value theorem is not valid because \([sX(s)]\) does not have a limit for some real \(s \geq 0\), i.e., at \(s = 3 \pm 2j\). For the same reason, \(x(t)\) is diverging (unbounded).

Figure S3.9b. *Simulation of X(s) for case b*)
$x(t)$ is oscillatory because the denominator of $[sX(s)]$ includes complex factors. See Fig. S3.9b.

c) $X(s) = \frac{16s + 5}{(s^2 + 9)(s + 3j)(s - 3j)}$ 

$x(0) = \lim_{s \to \infty} \left[ \frac{16s^2 + 5s}{(s^2 + 9)} \right] = 16$

Application of final value theorem is not valid because $[sX(s)]$ does not have a limit for real $s = 0$. This implies that $x(t)$ is not diverging, since divergence occurs only if $[sX(s)]$ does not have a limit for some real value of $s > 0$.

$x(t)$ is oscillatory because the denominator of $[sX(s)]$ is a product of complex factors. Since $x(t)$ is oscillatory, it is not converging either. See Fig. S3.9c

![Graph of $X(s)$ for case c](image)

**Figure S3.9c. Simulation of $X(s)$ for case c)**